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Viscous Shock Profiles for 2×2 Systems of Hyperbolic Conservation Laws with Quadratic Flux Functions

大阪電気通信大学・工学部　浅倉　史興 (Fumioki ASAKURA)
Osaka Electro-Communication Univ.
asakura@isc.osakac.ac.jp
国際基督教大学・教養学部　山崎　満 (Mitsuru YAMAZAKI)
International Christian Univ.
yamazaki@icu.ac.jp

1. Introduction

The purpose of this paper is to understand the shock wave structure of conservation laws that come from the extraction of petroleum.

An oil reservoir is a subsurface pool of hydrocarbons contained in porous rock formations. If the underground pressure of in the reservoir is sufficient, then the oil is naturally forced to the surface and extracted by valves on the well. This is called the *primary recovery* and usually about 20% of the oil in a oil reservoir can be extracted.

Over the lifetime of the well, the underground pressure will be insufficient to force the oil to the surface. *Secondary recovery* techniques increase the reservoir pressure by injecting water and gas (air or CO_2). Generally 25% to 35% of the oil in a oil reservoir can be extracted by primary and secondary recovery together.

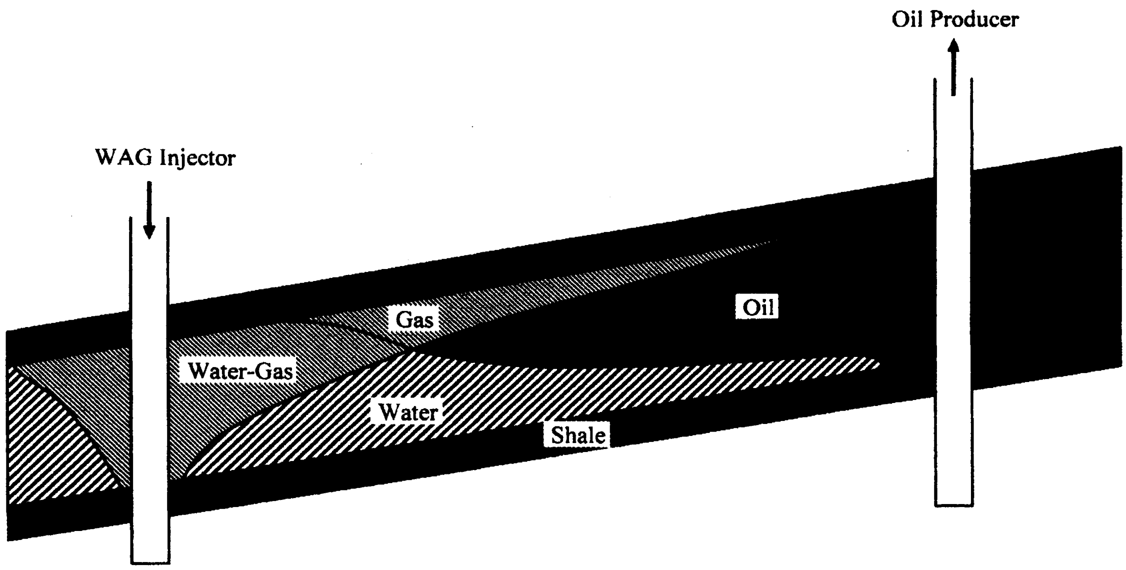

Figure 1: WAG Enhanced Oil Recovery (schematic picture)

Water-Alternating-Gas (WAG) Enhanced Oil Recovery: Although the water injection has good sweep efficiency, 40 to 60% of the original oil on-site is left behind at the end of the injection. The gas injection has good displacement efficiency but is an expensive operation. Hence the injection of gas after water followed by water and gas injection causes significant redistribution of fluids in the reservoir and will be more efficient than injection of water or gas alone.

Because of the gravity, three phases: oil, gas and water are separated from one another away from the WAG injector and it is only near the injector where three phase flow actually occurs. Mathematical structure of the three phase flows has been investigated by many authors (for example, Marchesin-Plohr [7], Medeiros [8], Schaeffer-Shearer [10]) and, in this paper, we shall confine ourselves particularly to their shock wave structure.

Stone's Model: In order to simplify the three-phase flow in a porous medium, we neglect the gravity and assume that the medium is homogeneous and the flow is incompressible and immiscible. Let us denote:

	water	gas	oil
Volume Fractions:	$s_W = u$	$s_G = v$	$s_O = 1 - u - v$
Permeability Functions:	k_W	k_G	k_O
Fluid Viscosity:	μ_W	μ_G	μ_O
Fluid Velocity:	v_W	v_G	v_O
Pressure:	p_W	p_G	p_O

The relationship between the flow rate and the pressure gradient is expressed by *Darcy's Law*

$$v_i = -\frac{k_i}{\mu_i}\nabla p_i, \quad i = W, G, O.$$

It is usually assumed that the water and gas permeability functions depend only on the water and gas volume fraction

$$k_W = k_W(u), \quad k_G = k_G(v)$$

which is called *Stone's assumption*. We finally assume that the flow is one dimensional and the capillary pressure is negligible.

By using *relative permeability functions*

$$f(u) = \frac{k_W(u)}{\mu_W}, \quad g(v) = \frac{k_G(v)}{\mu_G} \quad h(u,v) = \frac{k_O(u,v)}{\mu_W}$$

the *mass conservation laws* are expressed in the form

$$\text{Water:} \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\frac{f(u)}{f(u)+g(v)+h(u,v)}\right] = 0, \tag{1}$$

$$\text{Gas:} \quad \frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left[\frac{g(v)}{f(u)+g(v)+h(u,v)}\right] = 0 \tag{2}$$

in $\Omega: \ 0 < u + v < 1, \ u, \ v > 0$ ([7],[8], [10]). These equations constitute a system of conservation laws that is discussed in this paper.

Hyperbolicity: We say that the system of equations (1) and (2) is *hyperbolic,* when the Jacobian matrix of the flux function has *real* eigenvalues $\lambda_1(U), \lambda_2(U)$ for any $U \in \Omega$. If, in particular, these eigenvalues are *distinct*: $\lambda_1(U) < \lambda_2(U)$, the system is called *strictly hyperbolic* at U. Corresponding right eigenvectors are denoted by $R_1(U), R_2(U)$ respectively. A state $U^* \in \Omega$ is called an *umbilic point*, if $\lambda_1(U^*) = \lambda_2(U^*)$ and the Jacobian matrix is diagonalizable, hence a scalar matrix. Marchesin, Paes-Leme, Schaeffer and Shearer have shown in [10].

Theorem 1 (Existence of Umbilic Point) *Assume that $h(u,v) = h(1-u-v)$ and $f(0) = g(0) = h(0) = 0$, $f''(u), g''(v), h''(w) > 0$. Then the system of equations* (1), (2) *is hyperbolic and has a unique umbilic point in Ω.*

After the change of unknown functions, we may assume that $U^* = O$ and $F(O) = O$. Thus we have the Taylor expansion of the flux function $F(U)$ near $U = O$:

$$F(U) = \lambda^* U + Q(U) + O(1)|U|^3$$

where $\lambda^* = \lambda_1(U^*) = \lambda_2(U^*)$ and $Q : \mathbf{R}^2 \to \mathbf{R}^2$ is a homogeneous quadratic mapping. After the Galilean change of variables: $x \to x - \lambda^* t$, we observe that the system of equations (1) and (2) is reduced to

$$U_t + Q(U)_x = O, \quad (x,t) \in \mathbf{R} \times \mathbf{R}_+,$$

modulo higher order terms. By a change of unknown functions $V = S^{-1}U$ with a regular constant matrix S, we have a new system of equations $V_t + P(V)_x = 0$ with $P(V) = S^{-1}Q(SV)$. Hence we say that two quadratic mappings $Q_1(U)$ and $Q_2(U)$ are *equivalent*, if there is a constant matrix $S \in GL_2(\mathbf{R})$ such

$$Q_2(U) = S^{-1}Q_1(SU) \quad \text{for all} \quad U \in \mathbf{R}^2.$$

Schaeffer-Shearer [10] shows that every hyperbolic quadratic mapping $Q(U)$ with an isolated umbilic point $U = O$ is equivalent to

$$Q(U) = \frac{1}{2}\begin{pmatrix} au^2 + 2buv + v^2 \\ bu^2 + 2uv \end{pmatrix} = \frac{1}{2}\nabla C(U), \tag{3}$$

$$C(U) = \frac{1}{3}au^3 + bu^2v + uv^2. \tag{4}$$

where a and b are two real parameters satisfying $a \neq 1 + b^2$. For Stone's model, either

$$\textbf{Case I}: \ a < \frac{3}{4}b^2 \quad \text{or} \quad \textbf{Case II}: \ \frac{3}{4}b^2 < a < 1 + b^2.$$

A constant characteristic vector field $\Xi = {}^t(1, \xi)$ exists if and only if

$$\xi^3 + 2b\xi^2 + (a-2)\xi - b = -\Xi^{\perp}\nabla Q(\Xi)\Xi = 0$$

Three district (real) roots are denoted by μ_1, μ_2, μ_3 and *Medians* are defined by M_j : $v = \mu_j u, \ j = 1,2,3$. Medians play an important role in the following discussion.

Gomes' Paper [4] and our aim: M. E. S. Gomes has proved the existence of viscous shock profiles for shock waves in Case I by topological methods and also shown an example of compressive shock wave without viscous shock profiles. The aim of this paper is to complete her results by using both topological and analytical methods: existence of viscous profiles in Case I and II and general condition for non-existence of viscous profiles. We shall show in this paper only outline of proof and details will be published in Asakura-Yamazaki [2].

2. Undercompressive and Overcompressive Shock Waves

Rankine-Hugoniot condition: A jump discontinuity defined by

$$U(x,t) = \begin{cases} U_L & \text{for} \quad x < st, \\ U_R & \text{for} \quad x > st, \end{cases} \tag{5}$$

with a real constant s, is a piecewise constant weak solution to the the conservation laws (3), if and only if these quantities satisfy the *Rankine-Hugoniot condition*:

$$s(U_R - U_L) = Q(U_R) - Q(U_L). \tag{6}$$

The weak solution (5) satisfying (6) is often called a *shock wave* of speed s joining the state U_L, on the left, to the state U_R, on the right.

Compressive shock wave: The shock wave is said to be a *j-compressive* ($j = 1, 2$) if the speed satisfies the *Lax entropy conditions*:

$$\lambda_j(U_R) < s < \lambda_j(U_L), \;\; \lambda_{j-1}(U_L) < s < \lambda_{j+1}(U_R)$$

Here we adopt the convention $\lambda_0 = -\infty$ and $\lambda_3 = \infty$.

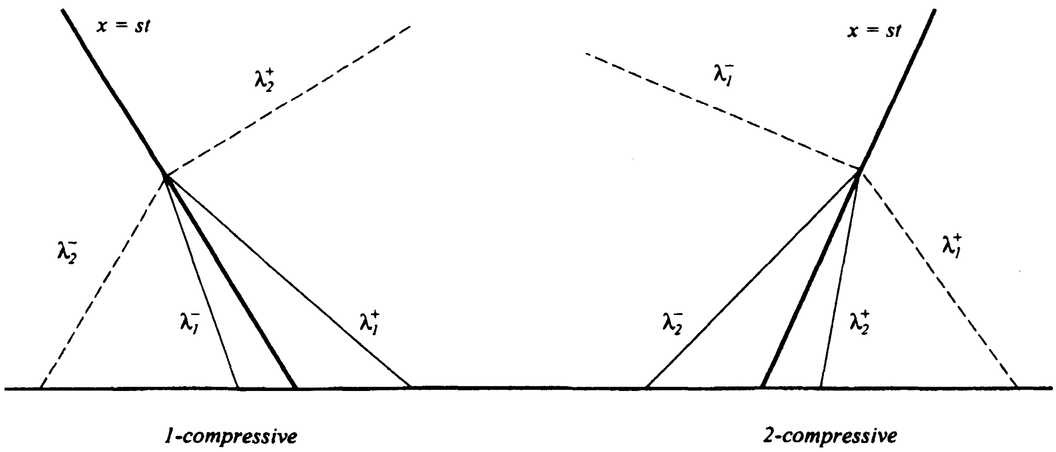

Figure 2: Compressive Shock waves

Undercompressive shock wave: *Undercompressive* if s satisfies

$$\lambda_1(U_R) < s < \lambda_2(U_R), \; \lambda_1(U_L) < s < \lambda_2(U_L)$$

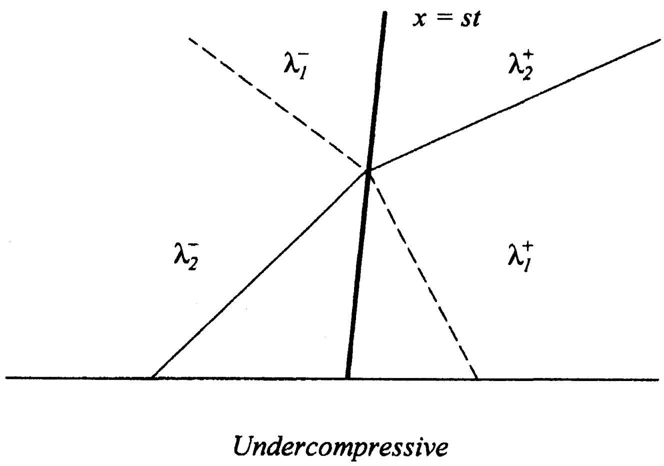

Figure 3: Undercompressive Shock wave

Overcompressive shock wave: *Overcompressive* if s satisfies

$$\lambda_1(U_R) < s < \lambda_1(U_L), \; \lambda_2(U_R) < s < \lambda_2(U_L)$$

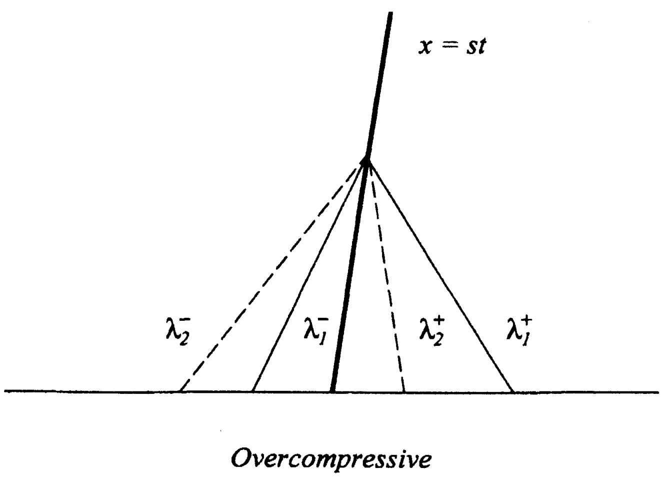

Figure 4: Overcompressive Shock wave

Stability and Admissibility of Shock Waves: It is generally believed

- *Compressive shock waves* are generally stable and admissibility is independent of diffusion matrices in a generic class.
- *Undercompressive shock waves* are stable with additional (kinetic) condition and admissibility depends on diffusion matrices.
- *Overcompressive shock waves* are generally unstable.

Admissibility is defined in next section.

3. Viscous Shock Profiles

Admissibility: The jump discontinuity is said to be *admissible* if there exists a travelling wave solution $U_\epsilon(x,t) = \widehat{U}(\frac{x-st}{\epsilon})$ to the parabolic system

$$U_t + Q(U)_x = \epsilon U_{xx}, \quad \epsilon > 0 \tag{7}$$

satisfying $U_\epsilon(+\infty, t) = U_R$, $U_\epsilon(-\infty, t) = U_L$. The vector function $\widehat{U} = \widehat{U}(\xi)$ is called a *viscous shock profile.*

Differential Equations and Vector Field: By integrating (7), $\widehat{U}(\xi)$ satisfies a system of nonlinear differential equations

$$\begin{aligned} \frac{d\widehat{U}}{d\xi} &= -s(\widehat{U} - U_L) + F(\widehat{U}) - F(U_L) \\ &= X_s(U, U_L) \end{aligned}$$

Note that U_L is a critical point of $X_s(U, U_L)$ and by Rankine-Hugoniot condition U_R is also a critical point. Since the flux functions has a potential $C(U)$, by setting

$$\phi_s(U_L, U) = C(U) - \nabla C(U_L) \cdot (U - U_L) - s|U - U_L|^2,$$

the differential equations turn out to be

$$\frac{d\widehat{U}}{d\xi} = \frac{1}{2}\nabla\phi_s(U_L, \widehat{U}). \tag{8}$$

Hence the admissibility is equivalent to the existence of solution of this equations satisfying the boundary conditions at infinity:

$$\lim_{\xi\to-\infty} \widehat{U}(\xi) = U_L, \ \lim_{\xi\to\infty} \widehat{U}(\xi) = U_R$$

or to finding flow connecting two *critical points* U_L and U_R of the vector field $\nabla\phi_s(U_L, U)$ (or ϕ_s equivalently).

4. Existence of Viscous Shock Profiles

Critical Points: Classification of compressive, undercompressive and overcompressive shock waves corresponds to that of critical points:

shock wave	U_L	U_R
1-compressive	node (repeller)	saddle
2-compressive	saddle	node (attractor)
undercompressive	saddle	saddle
overcompressive	node (repeller)	node (attractor)

There are at most four critical points in the finite plane (intersection of two conics). In four critical point case:

Case I: one node and three saddles [4], Case II: two nodes and two saddles [1]

Saddle-Saddle Connection: Flow of a saddle-saddle connection lies on $M_j,\ j = 1,2,3$ ([3],[4]). If $U_L \in M_j,\ j = 1,2,3$, The equation of viscous shock profile turns out to be the Burgers equation

$$\frac{du}{d\xi} = \frac{b+2\mu_j}{2\mu_j}(u-u_1)(u-u_L),\ u_1 = -u_L + \frac{2\mu_j}{b+2\mu_j}s.$$

By direct computations we have

Theorem 2 ([2]) *Undercompressive shocks with viscous profile exist only on $M_1 \cup M_2 \cup M_3$ in Case I and on $M_1 \cup M_3$ in Case II. Overcompressive shocks with viscous profile exist only on M_2 in Case II.*

Existence of Viscous Profiles: If there are no saddle-saddle connection, the connection problem is settled as the following:

Theorem 3 ([2], **Case I**) *If U_L is a node, then for each single saddle point there exists a viscous shock profile between U_L and the saddle point.*

Theorem 4 ([2], **Case II**) *Two nodes consist of one attractor and one repeller. If U_L is a node, then for each of two saddle points, there exists a viscous shock profile shock profile between U_L and the saddle point. Moreover there exist infinitely many viscous shock profiles from the repeller to the attractor.*

Proof of the above both theorems is based on a generalization of the first theorem of Morse to non-compact level sets: if $|\nabla\phi_s(U,U_L)|^2 \geq m$ for any $U \in \phi_s^{-1}[p,q]$, then

$$\phi_s^{-1}[p,q] = \bigcup_{U_p \in \phi_s^{-1}(p)} I(U_p): \quad \text{Morse foliation,}$$

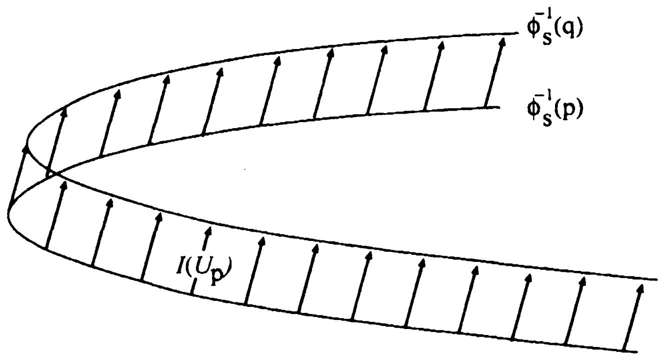

Figure 5: Morse Foliation

where $I(U_p)$: integral curves of the equation (8) connecting $U_p \in \phi_s^{-1}(p)$ and a certain point on the level set $\phi_s^{-1}(q)$.

Case I: We may assume that U_L is a repeller. Figure 7 to 9 show nine level curves of ϕ_s for $a = 0.5, b = 1, s = -3.5$ and $U_L = {}^t(1,\ 1)$. Let ε be a positive small constant. The level set $\{\phi_s = \varepsilon\}$ is composed of a small closed curve enclosing U_L and three unbounded regular curves (Fig 7: $\phi_s = 10.00$). Suppose that a critical point U_1 exists on the level set $\{\phi_s(U) = p_1\}$, (Fig.7: $p_1 = 25.88$) such that there is no critical point in $\{\varepsilon \le \phi_s(U) \le p_1 - \varepsilon\}$. By the Morse lemma , we find that $\phi_s^{-1}[\varepsilon, p_1 - \varepsilon]$ is a Morse foliation. When the level curve meets a critical point for $\phi_s(U) = p_1$, an integral curve connects two critical points (Fig.7: $\phi_s = 25.88$). Repeating this argument, we have three trajectories connecting critical points (Fig 8, 9).

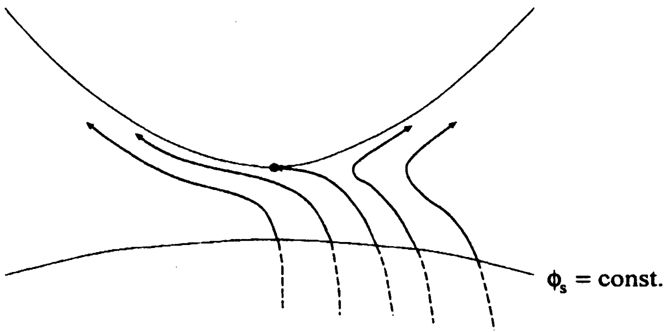

Figure 6: Flow at a Critical Point

Case II: We may assume that U_L is a repeller. Figure 10 to 12 show nine level curves of ϕ_s for $a = 1.5, b = 1, s = -1$ and $U_L = {}^t(1,\ 1)$. The level set $\{\phi_s = \varepsilon :\ \text{small}\}$ is composed of a small closed curve enclosing U_L and a single unbounded regular curves in this case (Fig 10: $\phi_s = 0.150$). Suppose that the first critical point U_1 exists on the level set $\{\phi_s(U) = p_1\}$, (Fig.10: $p_1 = 0.800$) such that there is no critical point in $\{\varepsilon \le \phi_s(U) \le p_1 - \varepsilon\}$. By the same argument as above, we find a trajectory connecting U_L and the first critical point (Fig 10: $\phi_s = 0.800$). Repeating this argument (Fig 11:$\phi_s = 1.000$ to 12:$\phi_s = 20.00$), we have the second trajectory.

Above the second critical point, we have a closed curve and a single unbounded curve (Fig 12: $\phi_s = 21.00, 27.00$). Since the closed curve encloses an attractor, we conclude that there are infinitely many trajectories issuing from U_L and drawn into the attractor.

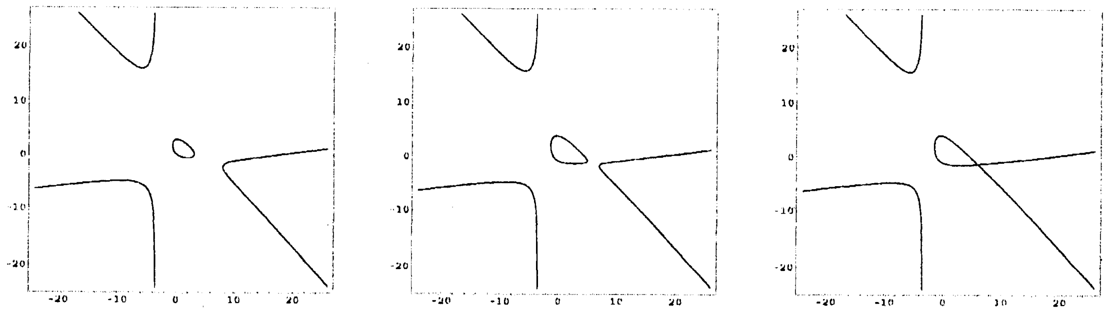

Figure 7: $\phi_s = 10.00,\ 23.00,\ 25.88$

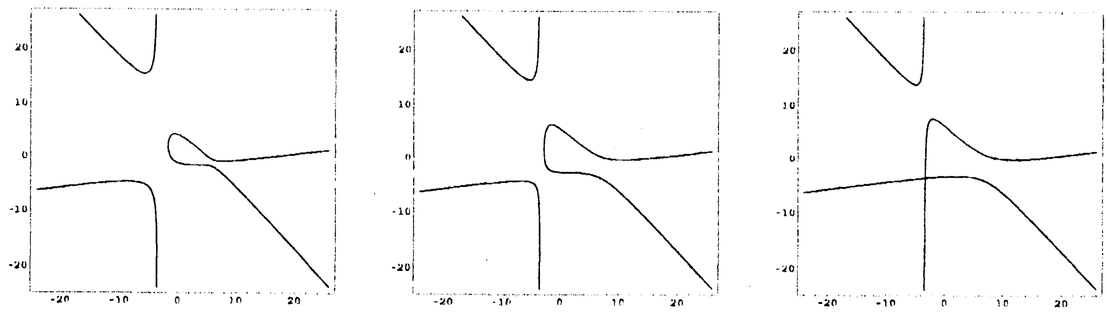

Figure 8: $\phi_s = 30.00,\ 65.00,\ 85.55$

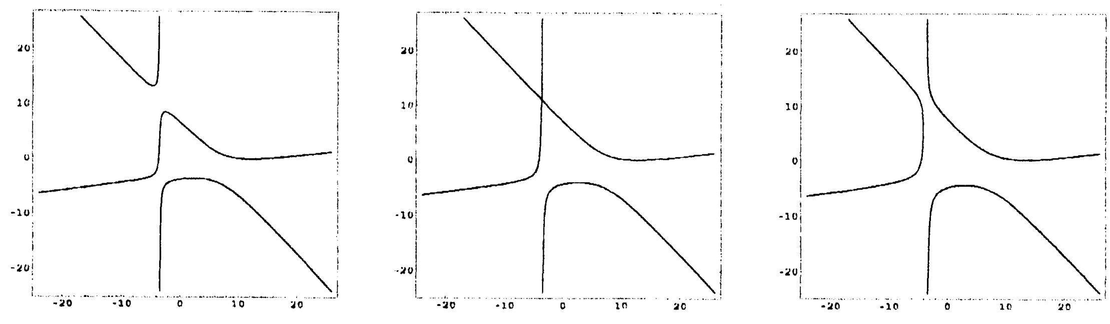

Figure 9: $\phi_s = 100.0,\ 118.5,\ 140.0$

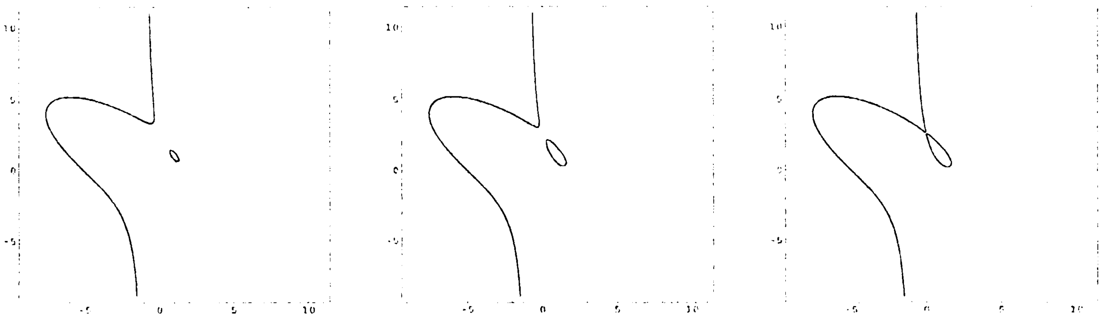

Figure 10: $\phi_s = 0.150,\ 0.600,\ 0.800$

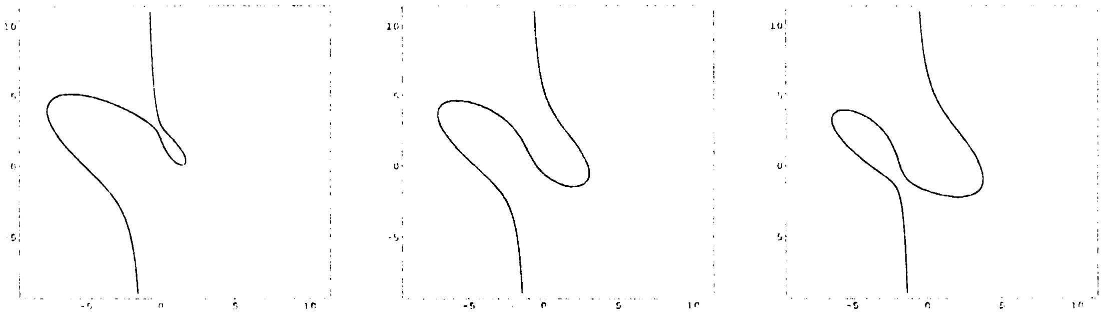

Figure 11: $\phi_s = 1.000,\ 10.00,\ 19.00$

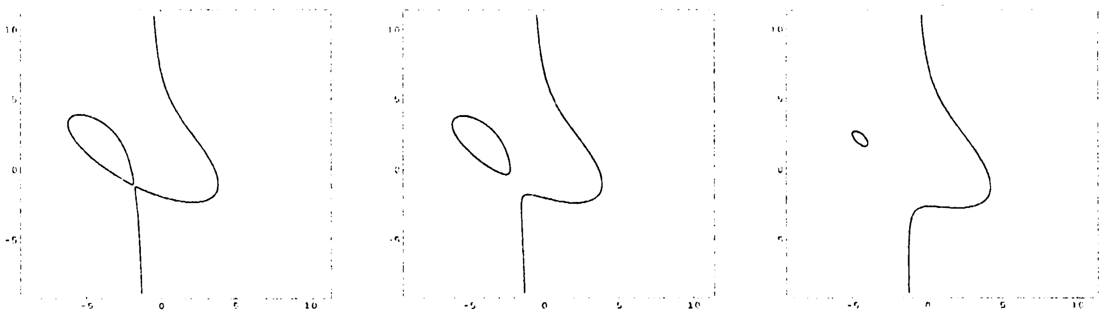

Figure 12: $\phi_s = 20.00,\ 21.00, 27.00$

5. Compressive Shock without Viscous Shock Profile

Liu-Oleinik Condition: Let us denote: $\mathcal{H}(U_L)$: $U = U(\xi; U_L)$ the Hugoniot curve issuing from U_L; $s(\xi)$: the shock speed at $U(\xi)$; $U_R = U(\xi_1)$. We say that U_L and U_R satisfy the (strict) *Liu-Oleinik condition* if $s(\xi_1) < s(\xi)$ for all $0 \leq \xi < \xi_1$ ([9]).

For strictly hyperbolic systems, as long as U_R is sufficiently close to U_L, there exists a viscous shock profile connecting these states if and only if they satisfy the Liu-Oleinik condition ([6]).

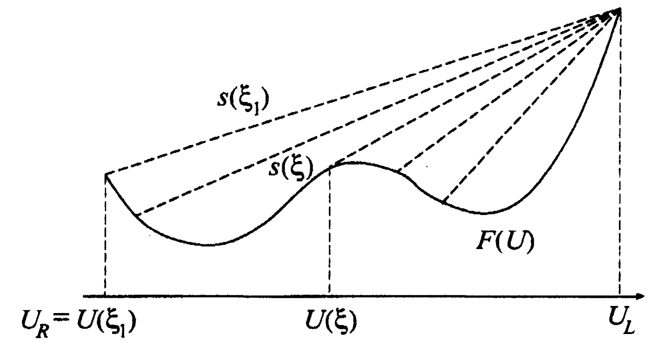

Figure 13: Liu-Oleinik Condition

State U_L on a Median (Case I): In this case, the Hugoniot curves are composed of the median and a hyperbola, and their intersection points are U_L(first bifurcation point) and U_* (second bifurcation point). We can deduce by Theorem 2 that there is a saddle-saddle connection (Fig. 14: left).

Theorem 5 ([2]) *Suppose that the medians and the inflection curves intersect only at the origin O and that $U_L \in M_j \setminus \{O\}$ $(j = 1, 2, 3)$. Then there exists one branch $\mathcal{H}_*$ of the hyperbola $\mathcal{H}_j(U_L)$ issuing from U_* such that the state U_L, on the left, can be joined to any state $\in \mathcal{H}_*$ sufficiently close to U_*, on the right, by an inadmissible, compressive Liu-Oleinik shock. In this case, there exists a saddle-saddle connection along M_j.*

Outline of proof: Let $U_L \in M_1$ and $u_L > 0$. We find by direct computation that the 2-shock curve issuing from U_L is composed of the segment $\overline{U_L U_*}$ and one branch of the hyperbola $\mathcal{H}_j$ issuing from U_*.

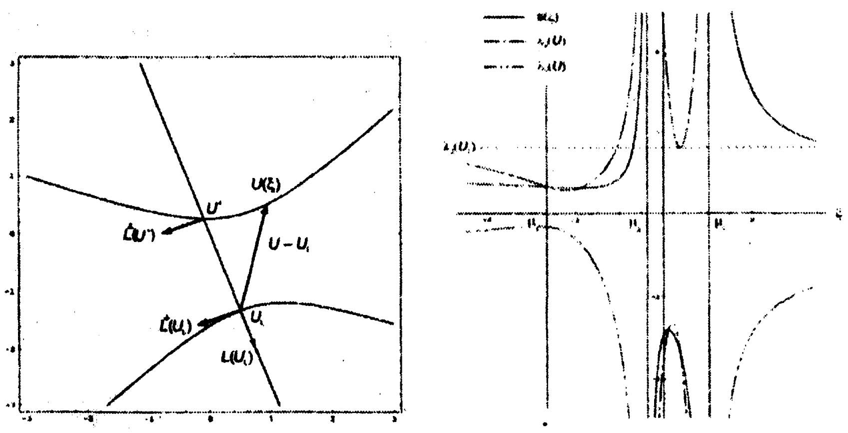

Figure 14: Hugoniot Locus and Shock Speed

Since $s = \lambda_2$ and $\dot{s} \neq \dot{\lambda}_2$ at U_*, one branch of hyperbola containing U_* is a compressive branch of the 2-shock curve. Hence by choosing the shock speed s close to s_* we have a 2-compressive shock connecting U_L and a state U_R that is close to U_*. As we have noticed there is a saddle-saddle connection from U_L to a certain state $U_1 \in M_1$

which is close to U_*, hence close to U_R. Thus we conclude from the configuration of trajectories that it is impossible.

Figure 14 is the Hugoniot curves and the graph of shock speed for $a = 0.1$, $b = 1$, $U_L = {}^t(0.5, 0.5\mu_1)$, $\mu_1 = -2.65004$, $\mu_2 = -0.369954$, $\mu_3 = 1.02$, $U_* = {}^t(-0.0484773, 0.128465)$, $s_* = 0.653999$; the parameter of the shock speed is $\xi = \frac{v-v_L}{u-u_l}$. In the graph of shock speed, s decreases from $\lambda_2(U_L)$ to λ_* along $\xi = \mu_1$. then the compressive blanch goes to the right. It is clear from this figure, the Liu-Oleinik condition actually holds. Figure 15 shows the level curves of the potential function for

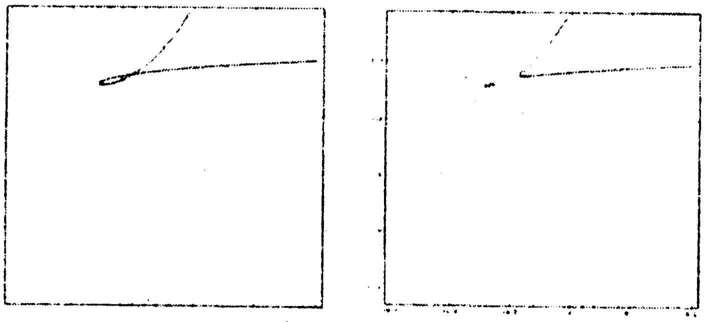

Figure 15: $s = 0.646$, $\phi_s = 0.4730785$ (left), $\phi_* = 0.473086$ (right)

$s = 0.646$. The left figure: $\phi_s = 0.4730785$ shows the connection of U_L and a certain state on the median M_1, hence the existence of an undercompressive shock wave. The right one shows a small closed level curve that encloses an attractor.

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